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Stability of Nonuniform Cracked Bars Under Arbitrarily Distributed Axial Loading

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This study is concerned with the stability analysis of nonuniform bars with arbitrary number of cracks and with arbitrary distribution of flexural stiffness or arbitrarily distributed axial loading. The homogeneous solutions of the governing differential equation for buckling of a nonuniform uncracked bar are derived for several important cases. A model of massless rotational spring is adopted to describe the local flexibility induced by cracks. Then a new approach that combines the exact buckling solution of a nonuniform uncracked bar, the model of massless rotational spring, and the transfer matrix method is presented for the title problem. The main advantage of the proposed method is that the eigenvalue equation for buckling of a nonuniform bar with an arbitrary number of cracks, arbitrary distribution of flexural stiffness, or arbitrarily distributed axial loading can be conveniently determined from a second-order determinant. The decrease in the determinant order as compared with other methods leads to significant savings in the computational effort. A numerical example is given to illustrate the reliability of the proposed approach through comparisons with numerical solutions and to study the effect of cracks on the stability of a nonuniform bar.

Nomenclature

a_i	=	depth of the ith crack
C_i	=	flexibility of the ith rotational spring
C_{i1}, C_{i2}	=	constants of integration
${}_1F_2(a_1; a_2; a_3; a_4)$	=	hypergeometric series
$f()$	=	flexibility function
h_i	=	height of cross section at x_i
$J_v(x)$	=	first Bessel function of the vth order
$K_i(x)$	=	flexural stiffness at x in the ith segment bar
K_{ui}	=	translational spring constant at x_i
$K_{\varphi i}$	=	rotational spring constant at x_i
$M_i(x)$	=	bending moment at x in the ith segment bar
N_{cr}	=	critical buckling force
$N_i(x)$	=	axial force at x in the ith segment bar
$Q_i(x)$	=	shear force at x in the ith segment bar
$S_{iN}(x)$	=	particular solutions of the governing equation
$S_{i1}(x)$, $S_{i2}(x)$	=	homogeneous solutions of the governing equation
$[T_i]$	=	transfer matrix for the ith segment
x, y	=	Cartesian coordinates with origin at the left end of a beam
$Y_v(x)$	=	the second Bessel function of the vth order
$\theta_i(x)$	=	rotation at x in the ith segment bar
$\Phi(a_1, a_2; a_3)$	=	Φ function

I. Introduction

DURING the past three decades, lightweight materials have been extensively used in civil, mechanical, aerospace, and other industrial fields; therefore, buckling problems of axially compressed structural members have become of increasing importance. Because the past studies on the problems of this type are quite extensive, only several selected research works on the buckling and vibration of cracked structural members are cited.

Cracks that are found to often exist in structural members certainly lower the structural integrity and should be considered in stability analysis for cracked structures. Anifantis and Dimarogonas[1] studied the stability of columns with single crack subjected to follower and vertical loads. Chen and Chen[2] investigated the vibration and stability of cracked rotating blades. However, these studies have been confined to uniform beam with cracks. The vibration and stability of a nonuniform Timoshenko beam has been studied by Takahashi,[3] but only one crack on the beam was considered in his study.

The vibration of cracked beams has been studied by many researchers. Rizos et al.[4] developed an approach for vibration analysis of a cracked beam. Their approach leads to a system of $(4n+4)$ equations for establishing the eigenvalue equation in the case of n cracks inside a uniform beam. An improved analytical method for calculating natural frequencies of a uniform beam with an arbitrary number of cracks was proposed by Shifrin and Ruotolo.[5] This procedure was presented based on the use of massless rotational spring to describe the local flexibility induced by cracks and, as a main feature, leads to a system of $(n+2)$ linear equations for determining the eigenvalue equation for a uniform beam with n cracks.

A model of massless rotational spring has been widely used to describe the local flexibility induced by cracks for studying vibration and stability problems of cracked structures (e.g., Refs. 5 and 6). The stability problem of a cracked structure can be thus simplified as that of the corresponding uncracked structure with massless rotational springs. A review of technical literature dealing with the buckling problem of a nonuniform structural member with elastic spring supports under variably distributed axial forces indicates that generally the authors of the previous studies have directed their investigations to special functions for describing the distributions of flexural stiffness and axial forces in order to derive analytical solutions (e.g., Refs. 7–10). It is also revealed that the analytical solution for buckling of general elastically restrained nonuniform structural members with arbitrary distribution of flexural stiffness or under arbitrarily distributed axial loading has not been proposed in the literature.

An attempt is made to present an analytical approach for stability analysis of a nonuniform bar with an arbitrary number of cracks and with classical and nonclassical boundary conditions. In this paper the function for describing the distribution of flexural stiffness $K(x)$ of a nonuniform bar is an arbitrary one, and the distribution of distributed axial loading $N(x)$ acting on the bar is expressed as a function of $K(x)$ and vice versa. The homogeneous solutions of the governing differential equation for buckling of a nonuniform uncracked bar are given for several important cases first. A model of massless rotational spring is adopted to describe the local flexibility induced

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by cracks. Then a new approach that combines the exact buckling solution of a nonuniform uncracked bar, the model of massless rotational spring, and the transfer matrix method is presented to establish the eigenvalues equation for buckling of a nonuniform cracked bar with classical and nonclassical boundary conditions. The main advantage of the proposed method is that the eigenvalue equation for buckling of a nonuniform bar with an arbitrary number of cracks, arbitrary distribution of flexural stiffness, or arbitrarily distributed axial loading can be conveniently determined from a second-order determinant. As a consequence, because of the decrease in the determinant order as compared with previously developed procedures (e.g., Refs. 4 and 5), the computational time required by the present method for solving the title problem can be reduced significantly. The numerical example shows that the results obtained from the present method are in good agreement with those determined from the finite element method (FEM), thus verifying the accuracy and reliability of the proposed method. It is also shown in the numerical example that the effect of cracks on the critical buckling force of a nonuniform bar significantly depends on the numbers, depths, and locations of cracks.

Although the stability problem considered in this paper might also be solved by approximated methods or numerical techniques (e.g., FEM), the present exact solutions can provide adequate insight into the physics of the problem, and the proposed analytical method can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximated or numerical solutions. Therefore, it is always desirable to present the exact solutions and the analytical method to such problems.

II. Buckling of Cracked Bar with Classical Boundary Conditions

A nonuniform bar with length L and with n cracks is shown in Fig. 1. It is assumed that the n cracks are located at sections $x_1, x_2, \ldots, x_n$ such that $0 < x_1 < x_2 < \cdots < x_n < L$. The bending moments of the bar under buckling are denoted by $M_i(x)$ on the interval $x_{i-1} < x < x_i$, where $i = 1, 2, \ldots, n+1$, $x_0 = 0$, and $x_{n+1} = L$. The effect of the ith crack is idealized as a rotational spring,[5,6] as shown in Fig. 1. Figure 2 shows the model of massless rotational spring to represent local flexibility induced by a crack in detail. The open-typed cracks considered in this study are traverse to the axis of the bar, and the depths of the cracks vary from 0 up to the height of the bar. For the sake of simplifying the analysis of the title problem, the interaction among the cracks in the bar is neglected in this study.

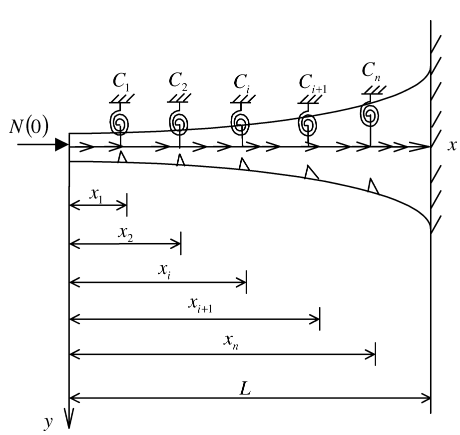

Fig. 1 Nonuniform cracked bar.

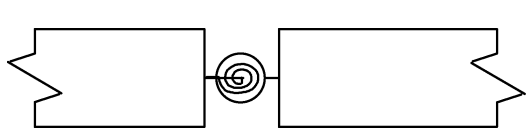

Fig. 2 Model of massless rotational spring to represent local flexibility induced by a crack.

It can be seen from Fig. 1 that the entire bar is divided into $(n+1)$ segments. The equation for buckling of each segment with nonuniform cross section and under arbitrary distribution of axial loading is[11−13]

$$\frac{\mathrm{d}^2 M_i(x)}{\mathrm{d}x^2} - \frac{1}{N_i(x)}\frac{\mathrm{d}N_i(x)}{\mathrm{d}x}\frac{\mathrm{d}M_i(x)}{\mathrm{d}x} + \frac{N_i(x)}{K_i(x)}M_i(x) = \frac{C_{i0}}{N_i(x)}\frac{\mathrm{d}N_i(x)}{\mathrm{d}x} \qquad (i = 1, \ldots, n+1), \quad (x_{i-1} < x < x_i) \tag{1}$$

where $M_i(x)$, $N_i(x)$, and $K_i(x)$ are the bending moment, axial force, and flexural stiffness of the ith segment bar at section x. It is assumed that $N_i(x)$ and $K_i(x)$ are continuous real-valued functions on the interval $[0, L]$; the subscript i in the expressions of $N_i(x)$ and $K_i(x)$ can be deleted in the rest part of this paper.

It is possible for each connection between two segment bars to introduce conditions that impose continuity for bending moment and shear force, respectively [see Eq. (2)]; moreover, the last equation in Eq. (2) introduces a discontinuity into the rotation of the bar axis, by imposing equilibrium between transmitted bending moment and rotation of the rotational spring representing the crack[5]:

$$\begin{aligned} &M_i(x_i) = M_{i+1}(x_i) \\ &Q_i(x_i) = Q_{i+1}(x_i) \\ &\theta_{i+1}(x_i) - \theta_i(x_i) = -C_i\frac{M_i(x_i)}{K(x_i)}, \qquad i = 1, 2, \ldots, n \end{aligned} \tag{2}$$

where C_i is the flexibility of the ith rotational spring, which is a function of the depth of the ith crack and the height of the cross section at $x = x_i$ of the bar.

For one-sided crack C_i can be expressed as[4]

$$C_i = h_i f(\xi_i) \tag{3}$$

where ξ_i can be determined by

$$\xi_i = a_i/h_i \tag{4}$$

in which a_i is the depth of the ith crack. $f(\xi_i)$ is called as flexibility function determined from the strain energy density function and is given by[4]

$$f(\xi_i) = 9.954\xi_i^2 - 21.117\xi_i^3 + 87.541\xi_i^4 - 199.010\xi_i^5 + 410.626\xi_i^6 - 678.407\xi_i^7 + 919.512\xi_i^8 - 769.664\xi_i^9 + 355.830\xi_i^{10} \tag{5}$$

The case of two-sided cracks can be considered similarly.[5] As is well known, the general solution of Eq. (1) can be obtained by means of the Lagrange method in the form

$$M_i(x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x) + C_{i0}S_{\mathrm{iN}}(x) \tag{6}$$

where

$$S_{\mathrm{iN}}(x) = -S_{i1}(x)\int \frac{S_{i2}(x)N'(x)}{D_i(x)}\,\mathrm{d}x + S_{i2}(x)\int \frac{S_{i1}(x)N'(x)}{D_i(x)}\,\mathrm{d}x \tag{7}$$

$$D_i(x) = N(x)\lfloor S_{i1}(x)S'_{i2}(x) - S_{i2}(x)S'_{i1}(x)\rfloor \tag{8}$$

$S_{i1}(x)$ and $S_{i2}(x)$ are the homogeneous solutions of Eq. (1), $S_{\mathrm{iN}}(x)$ is a particular solution of Eq. (1), respectively, and the primes indicate differentiation with respect to x.

It is evident that the key step for solving Eq. (1) is to determine the homogeneous solutions, which are found for two important cases and given in Appendix A. In this paper we also consider that a general case that is the function for describing the distribution of flexural stiffness $K(x)$ of a nonuniform bar is an arbitrary one, and the distribution of axial distributed loading $N(x)$ acting on the bar is expressed as a function of $K(x)$ and vice versa. The homogeneous solutions for buckling of nonuniform bar with arbitrary distribution of flexural stiffness or arbitrarily distributed axial loading are derived and presented in Appendix B.

After $S_{i1}(x)$, $S_{i2}(x)$, and $S_{iN}(x)$ $(i = 1, 2, \ldots, n+1)$ are found, differentiating Eq. (6) gives the shear force of the ith segment bar as follows:

$$Q_i(x) = C_{i1}S'_{i1}(x) + C_{i2}S'_{i2}(x) + C_{i0}S'_{iN}(x) \tag{9}$$

Using the relation between the shear force $Q_i(x)$ and the slope $\theta_i(x)$, one has

$$\theta_i(x) = C_{i1}\frac{S'_{i1}(x)}{N(x)} + C_{i2}\frac{S'_{i2}(x)}{N(x)} + C_{i0}\frac{1 + S'_{iN}(x)}{N(x)} \tag{10}$$

The relation between the parameters $\theta_i(x_i)$, $M_i(x_i)$, and $Q_i(x_i)$ at the section $x = x_i$ and the parameters $\theta_i(x_{i-1})$, $M_i(x_{i-1})$, and $Q_i(x_{i-1})$ at the section $x = x_{i-1}$ of the ith segment bar can be expressed as

$$\begin{bmatrix} \theta_i(x_i) \\ M_i(x_i) \\ Q_i(x_i) \end{bmatrix} = [T_i] \begin{bmatrix} \theta_i(x_{i-1}) \\ M_i(x_{i-1}) \\ Q_i(x_{i-1}) \end{bmatrix} \tag{11}$$

where

$$[T_i] = [A_i(x_i)][A_i(x_{i-1})]^{-1} \tag{12}$$

$$[A_i(x)] = \begin{bmatrix} \dfrac{S'_{i1}(x_i)}{N(x_i)} & \dfrac{S'_{i2}(x_i)}{N(x_i)} & \dfrac{1 + S'_{iN}(x_i)}{N(x_i)} \\ S_{i1}(x_i) & S_{i2}(x_i) & S_{i3}(x_i) \\ S'_{i1}(x_i) & S'_{i2}(x_i) & S'_{i3}(x_i) \end{bmatrix} \tag{13}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at a section $(x = x_{i-1})$ to those at another section $(x = x_i)$ of the ith segment bar.

Equation (2) can be written in the matrix form as

$$\begin{bmatrix} \theta_{i+1}(x_i) \\ M_{i+1}(x_i) \\ Q_{i+1}(x_i) \end{bmatrix} = \begin{bmatrix} 1 & -\dfrac{C_i}{K(x_i)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_i(x_i) \\ M_i(x_i) \\ Q_i(x_i) \end{bmatrix} \tag{14}$$

Using Eqs. (14) and (11), one yields

$$\begin{bmatrix} \theta_{i+1}(x_i) \\ M_{i+1}(x_i) \\ Q_{i+1}(x_i) \end{bmatrix} = [T_{iC}] \begin{bmatrix} \theta_i(x_{i-1}) \\ M_i(x_{i-1}) \\ Q_i(x_{i-1}) \end{bmatrix} \tag{15}$$

in which

$$[T_{iC}] = \begin{bmatrix} 1 & -\dfrac{C_i}{K(x_i)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [T_i] \tag{16}$$

Equation (11) can be rewritten as

$$\begin{bmatrix} \theta_{i+1}(x_{i+1}) \\ M_{i+1}(x_{i+1}) \\ Q_{i+1}(x_{i+1}) \end{bmatrix} = [T_{i+1}] \begin{bmatrix} \theta_{i+1}(x_i) \\ M_{i+1}(x_i) \\ Q_{i+1}(x_i) \end{bmatrix} \tag{17}$$

Substituting Eq. (15) into Eq. (17), one obtains

$$\begin{bmatrix} \theta_{i+1}(x_{i+1}) \\ M_{i+1}(x_{i+1}) \\ Q_{i+1}(x_{i+1}) \end{bmatrix} = [T_{i+1}][T_{iC}] \begin{bmatrix} \theta_i(x_{i-1}) \\ M_i(x_{i-1}) \\ Q_i(x_{i-1}) \end{bmatrix} \tag{18}$$

The equation for the last segment bar can be established using Eqs. (11) and (17) repeatedly as follows:

$$\begin{bmatrix} \theta_{n+1}(x_{n+1}) \\ M_{n+1}(x_{n+1}) \\ Q_{n+1}(x_{n+1}) \end{bmatrix} = [T] \begin{bmatrix} \theta_1(x_0) \\ M_1(x_0) \\ Q_1(x_0) \end{bmatrix} \tag{19}$$

in which

$$[T] = [T_{n+1}][T_{nC}] \cdots [T_{1C}] \tag{20}$$

and $[T]$ has the form as

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{21}$$

The element T_{ij} of $[T]$ can be found from Eq. (20).

The eigenvalue equation for buckling of a nonuniform bar with cracks can be established by using Eq. (19) and the corresponding boundary conditions as follows:

1) A nonuniform cracked bar with clamped-free end supports (Fig. 1):

The boundary conditions for this case are

$$\begin{aligned} M_1(x_0) &= 0 \\ Q_{n+1}(x_{n+1}) &= 0 \\ \theta_{n+1}(x_{n+1}) &= 0 \end{aligned} \tag{22}$$

Substituting Eq. (22) into Eq. (19) results in

$$\begin{bmatrix} 0 \\ M_{n+1}(x_{n+1}) \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \theta_1(x_0) \\ 0 \\ Q_1(x_0) \end{bmatrix} \tag{23}$$

From the preceding equation one has

$$\begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix} \begin{bmatrix} \theta_1(x_0) \\ Q_1(x_0) \end{bmatrix} = 0 \tag{24}$$

Because $\theta_1(x_0) \neq 0$, $Q_1(x_0) \neq 0$, one obtains the eigenvalue equation as

$$T_{11}T_{33} - T_{13}T_{31} = 0 \tag{25}$$

2) A nonuniform cracked bar with hinged-hinged end supports:

The boundary conditions can be written as

$$\begin{aligned} &M_1(x_0) = 0 \\ &M_{n+1}(x_{n+1}) = 0 \\ &Q_{n+1}(x_{n+1}) - N(x_{n+1})\theta_{n+1}(x_{n+1}) = Q_1(x_0) - N(x_0)\theta_1(x_0) \end{aligned} \tag{26}$$

Using the boundary conditions, Eq. (26), and Eq. (19) leads to the eigenvalue equation as

$$T_{23}[T_{31} - N(x_{n+1}) + N(x_0)] - T_{21}[T_{33} - N(x_{n+1})T_{13} - 1] = 0 \tag{27}$$

3) A nonuniform cracked bar with hinged-clamped end supports:

If the left end of the bar is hinged and the right end is clamped, then the boundary conditions are given by

$$\begin{aligned} M_1(x_0) &= 0 \\ \theta_{n+1}(x_{n+1}) &= 0 \\ Q_{n+1}(x_{n+1}) &= Q_1(x_0) - N(x_0)\theta_1(x_0) \end{aligned} \tag{28}$$

These conditions result in the eigenvalue equation as

$$T_{11}[T_{33} + N(x_0)] - T_{13}(T_{31} - 1) = 0 \tag{29}$$

III. Buckling of Cracked Bars with Nonclassical Boundary Conditions

A nonuniform cracked bar with nonclassical boundary conditions is shown in Fig. 3. The general solutions of the ith segment bar can be expressed as

$$M_i(x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x) + C_{i0}S_{\mathrm{iN}}(x) \tag{30}$$

$$Q_i(x) = C_{i1}S'_{i1}(x) + C_{i2}S'_{i2}(x) + C_{i0}S'_{\mathrm{iN}}(x) \tag{31}$$

$$\theta_i(x) = C_{i1}\frac{S'_{i1}(x)}{N(x)} + C_{i2}\frac{S'_{i2}(x)}{N(x)} + C_{i0}\frac{1+S'_{\mathrm{iN}}(x)}{N(x)} \tag{32}$$

$$y_i(x) = C_{i1}\int\frac{S'_{i1}(x)}{N(x)}\,\mathrm{d}x + C_{i2}\frac{S'_{i2}(x)}{N(x)} + C_{i0}\int\frac{1+S'_{\mathrm{iN}}(x)}{N(x)}\,\mathrm{d}x + C_{i3} \tag{33}$$

The relation between the parameters $y_i(x_i)$, $\theta_i(x_i)$, $M_i(x_i)$, and $Q_i(x_i)$ at the section $x = x_i$ and the parameters $y_i(x_{i-1})$, $\theta_i(x_{i-1})$, $M_i(x_{i-1})$, and $Q_i(x_{i-1})$ at the section $x = x_{i-1}$ of the ith segment bar can be expressed as

$$\begin{bmatrix} y_i(x_i) \\ \theta_i(x_i) \\ M_i(x_i) \\ Q_i(x_i) \end{bmatrix} = [T_i]\begin{bmatrix} y_i(x_{i-1}) \\ \theta_i(x_{i-1}) \\ M_i(x_{i-1}) \\ Q_i(x_{i-1}) \end{bmatrix} \tag{34}$$

in which

$$[T_i] = [B_i(x_i)][B_i(x_{i-1})]^{-1}$$

$$[B_i(x)] = \begin{bmatrix} \int\frac{S'_{i1}(x)}{N(x)}\,\mathrm{d}x & \int\frac{S'_{i2}(x)}{N(x)}\,\mathrm{d}x & \int\frac{1+S'_{\mathrm{iN}}(x)}{N(x)}\,\mathrm{d}x & 1 \\ \frac{S'_{i1}(x)}{N(x)} & \frac{S'_{i2}(x)}{N(x)} & \frac{1+S'_{\mathrm{iN}}(x)}{N(x)} & 0 \\ S_{i1}(x) & S_{i2}(x) & S_{\mathrm{iN}}(x) & 0 \\ S'_{i1}(x) & S'_{i2}(x) & S'_{\mathrm{iN}}(x) & 0 \end{bmatrix} \tag{35}$$

Using the relations

$$\begin{aligned} y_{i+1}(x_i) &= y_i(x_i) \\ \theta_{i+1}(x_i) &= \theta_i(x_i) - C_i\frac{M_i(x_i)}{K(x_i)} \\ M_{i+1}(x_i) &= M_i(x_i) \\ Q_{i+1}(x_i) &= Q_i(x_i) \end{aligned} \tag{36}$$

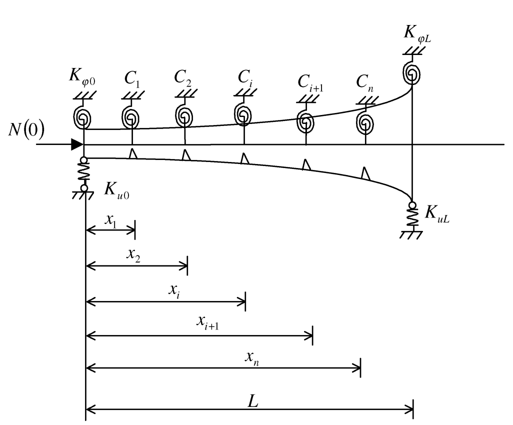

Fig. 3 Nonuniform cracked bar with nonclassical boundary conditions.

at the common interfaces of two neighboring segment bars leads to

$$\begin{bmatrix} y_{n+1}(x_{n+1}) \\ \theta_{n+1}(x_{n+1}) \\ M_{n+1}(x_{n+1}) \\ Q_{n+1}(x_{n+1}) \end{bmatrix} = [T]\begin{bmatrix} y_1(x_0) \\ \theta_1(x_0) \\ M_1(x_0) \\ Q_1(x_0) \end{bmatrix} \tag{37}$$

in which

$$[T] = [T_{n+1}][T_{\mathrm{nC}}]\cdots[T_{1C}]$$

$$[T_{\mathrm{iC}}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{C_i}{K(x_i)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}[T_i] \tag{38}$$

and $[T]$ has the following form:

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \tag{39}$$

If there are a rotational spring and a translational spring attached at $x = x_i$ (Fig. 4), then $[T_{\mathrm{iC}}]$ should be changed to

$$[T_{\mathrm{iK}}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{C_i}{K(x_i)} & 0 \\ 0 & -K_{\varphi i} & 1 & 0 \\ K_{\mathrm{ui}} & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

where $K_{\varphi i}$ and K_{ui} are the rotational spring constant and the translational spring constant (at $x = x_i$), respectively.

The boundary conditions for the case shown in Fig. 3 are

$$\begin{aligned} M_1(x_0) &= -K_{\varphi 0}\theta_1(x_0) \\ Q_1(x_0) &= K_{u0}y_1(x_0) + N(x_0)\theta_1(x_0) \\ M_{n+1}(x_{n+1}) &= K_{\varphi L}\theta_{n+1}(x_{n+1}) \\ Q_{n+1}(x_{n+1}) &= -K_{\mathrm{uL}}y_{n+1}(x_{n+1}) - N(x_{n+1})\theta_{n+1}(x_{n+1}) \end{aligned} \tag{41}$$

Substituting Eq. (41) into Eq. (37), one obtains

$$\begin{bmatrix} y_{n+1}(x_{n+1}) \\ \theta_{n+1}(x_{n+1}) \\ K_{\varphi L}\theta_{n+1}(x_{n+1}) \\ -K_{\mathrm{uL}}y_{n+1}(x_{n+1}) - N(x_{n+1})\theta_{n+1}(x_{n+1}) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \times \begin{bmatrix} y_1(x_0) \\ \theta_1(x_0) \\ -K_{\varphi 0}\theta_1(x_0) \\ K_{u0}y_1(x_0) + N(x_0)\theta_1(x_0) \end{bmatrix} \tag{42}$$

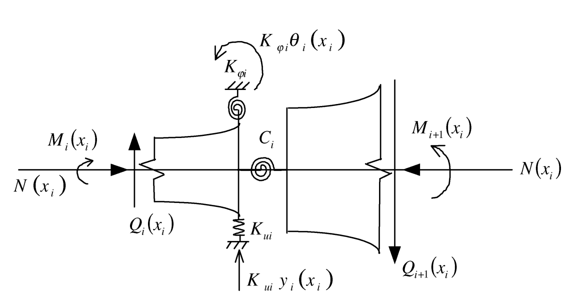

Fig. 4 Common interface of two neighboring step bars with a crack and springs.

From the preceding equation we obtain the eigenvalue equation as

$$P_{11}Q_{22} - P_{12}Q_{21} = 0 \tag{43}$$

where

$$\begin{aligned}
P_{11} &= K_{\varphi L}K_{\varphi 0} - T_{11}K_{\varphi 0} + T_{12}N(x_0) - K_{\varphi L}T_{32}N(x_0) + T_{13} \\
&\quad - K_{\varphi L}T_{33} - K_{\varphi L}T_{33} \\
P_{12} &= T_{12}K_{u1} - K_{\varphi L}K_{u1}T_{32} + T_{14} - K_{\varphi L}T_{44} \\
Q_{21} &= N(x_{n+1})N(x_0)T_{32} - N(x_{n+1})K_{\varphi 0}T_{31} + N(x_{n+1})T_{33} \\
&\quad - K_{\mathrm{uL}}K_{\varphi 0}T_{41} + N(x_0)K_{\mathrm{uL}}T_{42} + K_{\mathrm{uL}}T_{43} \\
Q_{22} &= N(x_{n+1})K_{u0}T_{32} + N(x_{n+1})T_{34} + K_{\mathrm{uL}}K_{u0}T_{42} + K_{\mathrm{uL}}T_{44}
\end{aligned} \tag{44}$$

IV. Numerical Example

A cantilever bar with cracks is shown in Fig. 1. It is assumed that

$$\begin{aligned}
&K(x) = K(0)(1+\beta x)^3, \quad N(x) = N(0)(1+\beta x)^2 \\
&\beta/L = 0.28, \quad x_1 = 0.4L, \quad x_2 = 0.6L, \quad x_3 = 0.8L, \quad n = 3 \\
&K(0) = 2\times 10^9\ \mathrm{N\cdot m^2} \\
&\xi_i = a_i/h_i = 0.3, \quad h_1 = 0.8896\ \mathrm{m}, \quad h_2 = 0.9344\ \mathrm{m}, \\
&h_3 = 0.9792\ \mathrm{m}
\end{aligned} \tag{45}$$

The procedure for determining the critical buckling force of the cracked cantilever bar is as follows:

1) Determination of the particular solutions of the homogeneous equation of Eq. (1): The distributions of $K(x)$ and $N(x)$ given in Eq. (45) belong to the case 1 considered in Appendix A, the particular solutions can be found from Eq. (A5) in Appendix A as follows:

$$\begin{aligned}
S_1(x) &= (1+\beta x)^{\frac{3}{2}} J_3\left[2\lambda(1+\beta x)^{\frac{1}{2}}\right] \\
S_2(x) &= (1+\beta x)^{\frac{3}{2}} Y_3\left[2\lambda(1+\beta x)^{\frac{1}{2}}\right]
\end{aligned} \tag{46}$$

2) Determination of the nonhomogeneous solution of Eq. (1): The nonhomogeneous solution of Eq. (1) can be found by using $S_1(x)$ and $S_2(x)$ given in Eq. (46) as follows:

$$\begin{aligned}
S_N(x) &= [(1+\beta x)/\beta]\,{}_1F_2(1;2;-1;-\xi^2/4) \\
\xi &= 2\lambda(1+\beta x)^{\frac{1}{2}}
\end{aligned} \tag{47}$$

where ${}_1F_2(a_1;a_2;a_3;a_4)$ is a hypergeometric series.

3) Determination of the flexibility of the ith rotational spring: $f(\xi_i)$ can be found from Eq. (5) as

$$f(\xi_i) = 0.7505, \qquad i = 1,\ 2,\ 3$$

$$C_1 = h_1 f(\xi_1) = 0.6677, \qquad C_2 = h_2 f(\xi_2) = 0.7013$$

$$C_3 = h_3 f(\xi_3) = 0.7349$$

4) Determination of the critical buckling force: Using Eqs. (46), (47), and (12) obtains $[T_i]$ $(i = 1, 2, 3, 4)$, then, using Eq. (20) yields $[T]$. The eigenvalue equation for this case is Eq. (25). Solving it results in the critical buckling force as

$$N_{\mathrm{cr}}^{(1)}(0) = 0.8103K(0)/L^2$$

The critical buckling force of the corresponding uncracked bar is found as

$$N_{\mathrm{cr}}(0) = 0.8760K(0)/L^2$$

$N_{\mathrm{cr}}^{(1)}(0)$ is 7.56% less than $N_{\mathrm{cr}}(0)$ because of the existence of the cracks inside the bar. If the depth of cracks is decreased from $0.3h_i$ to $0.2h_i$ and $0.1h_i$, respectively, then

$$N_{\mathrm{cr}}^{(2)}(0) = 0.8418K(0)/L^2$$

$$N_{\mathrm{cr}}^{(3)}(0) = 0.8627K(0)/L^2$$

$N_{\mathrm{cr}}^{(2)}(0)$ and $N_{\mathrm{cr}}^{(3)}(0)$ are 3.97 and 1.59% less than $N_{\mathrm{cr}}(0)$, respectively. If the locations of cracks are changed to the locations

$$x_1 = 0.5L, \quad x_2 = 0.7L, \quad x_3 = 0.9L, \quad \xi_i = 0.3 \qquad (i = 1, 2, 3)$$

then the critical buckling force is found to be

$$N_{\mathrm{cr}}^{(4)}(0) = 0.7980K(0)/L^2$$

$N_{\mathrm{cr}}^{(4)}(0)$ is 8.88% less than $N_{\mathrm{cr}}(0)$.

If the number of cracks is increased from three to six and the locations of the cracks are

$$\begin{aligned}
&x_1 = 0.4L, \qquad x_2 = 0.5L, \qquad x_3 = 0.6L \\
&x_4 = 0.7L, \qquad x_5 = 0.8L, \qquad x_6 = 0.9L \\
&\xi_i = 0.3 \qquad (i = 1, 2, \ldots, 6)
\end{aligned}$$

then the critical buckling force is found as

$$N_{\mathrm{cr}}^{(5)}(0) = 0.7796K(0)/L^2$$

$N_{\mathrm{cr}}^{(5)}(0)$is 11.07% less than $N_{\mathrm{cr}}(0)$.

To examine the accuracy and reliability of the proposed method, the FEM with cubic interpolation functions was also adopted to analyze this problem. The critical buckling forces of the bar for the cases just considered are determined using FEM as

$$\tilde{N}_{\mathrm{cr}}^{(1)}(0) = 0.8108K(0)/L^2, \qquad \tilde{N}_{\mathrm{cr}}^{(2)}(0) = 0.8422K(0)/L^2$$

$$\tilde{N}_{\mathrm{cr}}^{(3)}(0) = 0.8631K(0)/L^2, \qquad \tilde{N}_{\mathrm{cr}}^{(4)}(0) = 0.7984K(0)/L^2$$

$$\tilde{N}_{\mathrm{cr}}^{(5)}(0) = 0.7801K(0)/L^2$$

It is evident that all of the numerical results determined from FEM are in good agreement with those obtained from the present method, thus illustrating the proposed method is reliable. Meanwhile, it can be seen from the preceding results that the effect of cracks on the critical buckling force of a nonuniform bar significantly depends on the number, depths, and locations of the cracks. As the numbers and depths of cracks increase and the locations of cracks close to the fixed end of a cantilever bar, the critical buckling force of the bar decreases accordingly.

V. Conclusions

An efficient approach for stability analysis of a nonuniform bar with an arbitrary number of cracks and with classical and nonclassical boundary conditions is proposed. In this paper, the function for describing the distribution of flexural stiffness $K(x)$ of a nonuniform bar is an arbitrary one, and the distribution of axial distributed loading $N(x)$ acting on the bar is expressed as a function of $K(x)$ and vice versa. The homogeneous solutions of the governing differential equation for buckling of a nonuniform uncracked bar are derived first. A model of massless rotational spring is adopted to describe the local flexibility induced by cracks. Then a new approach that combines the exact buckling solution of a nonuniform uncracked bar, the model of massless rotational spring, and the transfer matrix method is presented to establish the eigenvalues equation for buckling of a nonuniform cracked bar with classical and nonclassical boundary conditions. The main advantage of the proposed method is that the eigenvalue equation for buckling of a nonuniform bar with an arbitrary number of cracks, arbitrary distribution of flexural stiffness, or under arbitrarily distributed axial loading can be conveniently determined from a second-order determinant. As a consequence, because of the decrease in the determinant order as compared with previously developed procedures (e.g., Refs. 4 and 5), the computational time required by the present method for solving the title problem can be reduced significantly. The numerical example shows that the results obtained from the present method are in good agreement with those determined from the finite element method, thus verifying the accuracy and reliability of the proposed method. It is also shown through the numerical example that the effect of cracks on the critical buckling force of a nonuniform bar significantly depends on the numbers, depths, and locations of cracks, and as the numbers and depths of cracks increase and the locations of cracks closes to the fixed end of a cantilever bar, the critical buckling force of the bar decreases accordingly.

Appendix A: Homogeneous Solutions of the Buckling Governing Equation for Two Cases

Obviously, the homogeneous solutions of the governing equation for buckling are dependent on the distributions of $K(x)$ and $N(x)$. For the following two types of distributions of $K(x)$ and $N(x)$, which describe many cases of ordinary structural members, the homogeneous solutions are found as follows.

Case 1:

$$K(x) = K(0)e^{-\beta(x/L)}, \qquad N(x) = N(0)e^{-b(x/L)} \tag{A1}$$

where β and b are parameters that can be determined by the values of $K(x)$ and $N(x)$ at the critical sections of the bar.

Substituting Eq. (A1) into Eq. (1) and setting

$$C_0 = 0, \qquad \xi = e^{(\beta-b)x/2L}, \qquad M = \xi^{v} z$$

lead to a Bessel equation of the vth order; the homogeneous solutions of Eq. (1) are as

$$\begin{aligned} S_1(x) &= e^{(\beta-b)vx/2L} J_v[\lambda e^{(\beta-b)x/2L}] \\ S_2(x) &= e^{(\beta-b)vx/2L} Y_v[\lambda e^{(\beta-b)x/2L}] \\ \lambda^2 &= \frac{4N(0)L^2}{K(0)(\beta-b)^2}, \qquad v = \frac{b}{b-\beta} \end{aligned} \tag{A2}$$

Case 2:

$$K(x) = K(0)(1+\beta x)^{b}, \qquad N(x) = N(0)(1+\beta x)^{c} \tag{A3}$$

where b and c are parameters that can be determined by the values of $K(x)$ and $N(x)$ at the critical sections of the bar.

For example, if the variations of width and height of a bar are described by $b(x) = b_0(1+\beta x)^m$ and $h(x) = h_0(1+\beta x)^n$, respectively, it is easily determined that $K(0) = b_0 h_0$ and $b = 3n + m$. If the width and height are linearly varied, that is, $m = n = 1$, then one has $b = 4$. If $b(x) = b_0 =$ constant and $h(x)$ is linearly varied, it is found that $b = 3$. These cases are often encountered in engineering practices.

Substituting Eq. (A3) into Eq. (1) and setting

$$C_0 = 0, \qquad \xi = 1 + \beta x, \qquad M = \xi^{(1+c)/2} z$$

result in a Bessel equation of the vth order; the homogeneous solutions of Eq. (1) are

$$\begin{aligned} S_1(x) &= (1+\beta x)^{(1+c)/2} J_v\left[\frac{\lambda}{a}(1+\beta x)^{a}\right] \\ S_2(x) &= (1+\beta x)^{(1+c)/2} Y_v\left[\frac{\lambda}{a}(1+\beta x)^{a}\right] \\ \lambda^2 &= \frac{N(0)}{K(0)\beta^2}, \qquad a = \frac{c-b+2}{2}, \qquad v = \frac{1+c}{c-b+2} \end{aligned} \tag{A4}$$

If $b = c + 2$, the homogeneous solutions of Eq. (1) can be easily obtained because the homogeneous equation is an Euler equation for this special case.

Appendix B: Homogeneous Solutions of the Buckling Governing Equation for More General Cases

The homogeneous solutions for two important cases are given in Appendix A. A general case is considered herein. That is, the function for describing the distribution of flexural stiffness $K(x)$ of a nonuniform bar is an arbitrary one, and the distribution of axial distributed loading $N(x)$ acting on the bar is expressed as a function of $K(x)$ and vice versa. We let

$$\begin{aligned} N(x) &= \text{arbitrary}, \qquad K(x) = N^{-1}(x)P^{-1}(\varsigma) \\ \varsigma &= \int N(x)\,\mathrm{d}x, \qquad M(x) = M(\varsigma) \end{aligned} \tag{B1}$$

or

$$\begin{aligned} K(x) &= \text{arbitrary}, \qquad N(x) = K^{-1}(x)P^{-1}(\varsigma) \\ \varsigma &= \int N(x)\,\mathrm{d}x, \qquad M(x) = M(\varsigma) \end{aligned} \tag{B2}$$

Substituting Eq. (B1) or Eq. (B2) into the homogeneous equation of Eq. (1) leads to

$$\frac{\mathrm{d}^2 M(\varsigma)}{\mathrm{d}\varsigma^2} + P(\varsigma)M(\varsigma) = 0 \tag{B3}$$

Obviously, it is easier to solve Eq. (B3) than to find the homogeneous solutions of Eq. (1). The solutions of Eq. (B3) are dependent on the expression of $P(\varsigma)$. Several important cases are considered and discussed as follows.

Case 1:

$$P(\varsigma) = a(1+b\varsigma)^{c} \tag{B4}$$

Substituting Eq. (B4) into Eq. (B3) and using the transformation

$$M = \eta^{v} z, \qquad v = 1/(c+2), \qquad \eta = (1+b\varsigma)^{1/2v} \tag{B5}$$

one obtains a Bessel equation of the order v. The particular solutions of Eq. (B3) are given by

$$\begin{aligned} S_1(\varsigma) &= (1+b\varsigma)^{\frac{1}{2}} J_v\left[\bar{\alpha}(1+b\varsigma)^{1/2v}\right] \\ S_2(\varsigma) &= (1+b\varsigma)^{\frac{1}{2}} Y_v\left[\bar{\alpha}(1+b\varsigma)^{1/2v}\right] \\ \bar{\alpha} &= (4a-b^2)/b^2 \end{aligned} \tag{B6}$$

If $c = 2$, Eq. (B3) becomes an Euler equation.

Case 2:

$$P(\varsigma) = ae^{b\varsigma} - c \tag{B7}$$

Substituting Eq. (B7) into Eq. (B3) and setting

$$\eta = e^{b\varsigma/2}$$

result in a Bessel equation of the order v. The particular solutions of Eq. (B3) are

$$\begin{aligned} S_1(\varsigma) &= J_v(\tilde{\alpha}e^{b\varsigma/2}) \\ S_2(\varsigma) &= Y_v(\tilde{\alpha}e^{b\varsigma/2}) \\ \tilde{\alpha} &= 4a/b^2, \qquad v^2 = 4c/b^2 \end{aligned} \tag{B8}$$

If $c = 0$, then $v = 0$.

Case 3:

$$P(\varsigma) = a\varsigma^{c-2} - b^2\varsigma^{2c-2} \tag{B9}$$

The particular solutions for this case are

$$S_1(\varsigma) = \varsigma^{(1-c)/2}\Phi\left(\frac{a}{2bc}, \frac{1}{2c}; \frac{2b}{c}\varsigma^c\right)$$

$$S_2(\varsigma) = \left(\frac{2b}{c}\varsigma^c\right)^{1-(1/2c)}\Phi\left(\frac{a}{2bc} - \frac{1}{2c} + 1, 2 - \frac{1}{2c}; \frac{2b}{c}\varsigma^c\right) \tag{B10}$$

where $\Phi(a_1,\ a_2;\ a_3)$ represents the Φ function.[14]

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